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A perturbative solution of the wave equation for a class of singular potentials

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Abstract : We employ the perturbation method developed by Muller, Muller-Kirsten and his coworkers to obtain the solution of the Schrodinger wave equation for a class of singular potentials described by

$$V(r) = \frac{a}{r^{\nu}} + \frac{b}{r^{\nu+1}} + \frac{c}{r^{\nu+2}} + \frac{d}{r^{\nu+3}},$$

where ν is an integer. First we extend the available perturbation results to the next higher order and in the process, we come across some correction terms which have not been accounted for in the previous analyses. We then, obtain the asymptotic expansions for energy eigenvalues and eigenfunctions for the above potentials. These asymptotic expansions provide highly convergent results in the close vicinity of the minima of the potential. For ν 1, the zero energy solutions of the wave equation using super symmetric quantum mechanics have also been discussed.

Keywords : Schrödinger wave equation, asymptotic expansion, singular potentials.

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1. Introduction

It is well known that the exact solution of the Schrödinger equation is possible only for a few choices of the potential (e.g. the potentials for harmonic oscillator and for Coulomb and $1/r^4$ interactions). For a large variety of other potentials which also occur frequently in different physical problems, one either uses an approximation method to obtain the solution (eigenvalues and eigenfunctions) of the wave equation, or solves the wave equation numerically. The analytic results, however, even if available in approximate form are important in their own right as compared to the corresponding better approximated numerical results. Furthermore the approximation methods, in spite of having some limitations in terms of their validity, are frequently used and sometimes pose serious problems for certain type

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of potentials such as singular potentials. These singular potentials have been studied for a long time (Predazzi and Regge 1962, Stroffolini 1971) and they still play an important role in the areas of molecular physics, particle and astrophysics (Barut and Kraus 1975, 1976, Barut 1980, Wald 1984).

Muller (1962, 1963, 1970), Muller-Kirsten and his coworkers (Muller-Kirsten and Vahedi-Faridi 1973, Aly *et al* 1975, Muller-Kirsten and Bose 1979 Muller-Kirsten 1980, Kaushal and Muller-Kirsten 1979, Kaushal 1979, Muller-Kirsten *et al* 1979, Muller-Kitsten and Sharma 1982) have developed a perturbation method and applied the same to a large class of potentials such as Yukawa, Gauss, power and logarithmic potentials. The scattering by a singular potential of the type g/r^4 has also been studied in this framework by Aly *et al* (1975). This method basically provides explicit solution of the wave equation in terms of asymptotic expansions and allows to generate higher order contributions symmetrically in terms of certain recurrence relations by considering the oscillator-like expansions around the minima of the potential. More recently, the large order behaviour (Muller-Kirsten and Sharma 1982) of this perturbation method, its superiority (Muller-Kirsten 1980, Boukema 1964a, 1964b) over WKB method and its comparison (Kaushal 1984) with large N-expansion method has also been investigated.

This perturbation method in conjunction with the WKB method provides the complete solution of the wave equation corresponding to a given potential. In fact, the perturbation solutions are valid in the close vicinity of the minima of the potential whereas the WKB solutions are valid in the regions far from the minima. The problem of normalization of the eigensolutions so obtained has been studied (Kaushal and Muller-Kirsten 1979) in detail with reference to an arbitrary power potential. This is done by considering the matching of the perturbative and WKB solutions in the regions of common validity. Interestingly, both these solutions are associated with one and the same eigenvalue i.e. the asymptotic eigenvalue expansion turns out to be the same in both the cases. However in either case, the terms in the asymptotic expansion have been computed only upto $1/h^2$ where h^4 is a measure of the curvature of the potential at the extremum point.

In the present work, we use this perturbation method to obtain the asymptotic expansions for energy eigenvalues and eigenfunctions for a class of singular potentials by

$$V(r)=\frac{a}{r^{\nu}}+\frac{b}{r^{\nu+1}}+\frac{c}{r^{\nu+2}}+\frac{d}{r^{\nu+3}}, \tag{1}$$

where ν is an integer. We extend the computation of higher order terms upto $1/h^4$ and find that $1/h^3$-term in the eigenvalue expansion also contributes to the eigenvalues. This somehow could not be noticed in earlier works. Though these correction terms are computed here with reference to the potential (1), their basic structure remains the same even for other potentials and their relative contribution depends on the values of the parameters of the potential.

Apart from the mathematical interest another motivation for undertaking this problem is that the potential (1) for $\nu=1$ arises in the studies of superpositronium as carried out by Barut and his coworkers (Barut and Kraus 1975, 1976, Barut 1980, Barut *et al* 1980). They obtained a Schrödinger-like equation involving the potential (1) with $\nu=1$ and solve the same rather numerically. The potential parameters a, b, c and d in this case, appear in terms of charges and magnetic moments of the two spin - $\frac{1}{2}$ particles. This potential (1) for $\nu=2, 3, 4$......also has a lot many applications in molecular physics particularly in the context of Van der Waals-type long range forces.

The arrangement of the paper is as follows : In the next section we briefly outline the underlying perturbation method for a corresponding one-dimensional problem. In Section 3, we explicitly compute the higher order terms upto $1/h^4$ in the energy eigenvalue expansion derived in Section 2. The generalization of the results to three-dimensions has been carried out in Section 4. Finally in Section 5, we discuss our main results and also offer the computation of zero-energy solutions using super symmetric quantum mechanics for a potential of the type (1).

2. Solution of the wave equation

For the complete solution of the problem one should find both perturbative and WKB solutions in the different regions of the extrema of the potential (1). Since the method has already been discussed and used (Kaushal and Muller-Kirsten 1979, Muller-Kirsten et *al* 1979) extensively, we give here only the necessary steps of the perturbation method. We first, obtain the solution of the one-dimensional Schrödinger equation ($2m=\hbar=1$, $\lambda^2=E$)

$$\left[-\frac{d^2}{dx^2}+V(x)-\lambda^2\right]\psi=0, \tag{2}$$

with the potential

$$V(x)=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\frac{d}{x^4}, \tag{3}$$

and later we shall discuss the possible generalization of the results so obtained. We assume that the first term in eq. (3) can be treated as perturbation (i.e. a is small).

We expand the potential (3) around its minima points $\bar{x}$ (in fact, one should label $\bar{x}$ with a subcript corresponding to different minima, but we drop the same for the sake of clarity in the subsequent expressions) as

$$V(x)=V(\bar{x})+\sum_{i=2}^{\infty}\frac{(x-\bar{x})^i}{i\,!}V^{(i)}(\bar{x}), \tag{4}$$

where $\bar{x}$ is determined in terms of the minima points x_0 of the function $\tilde{V}(x) = V(x) - a/x = b/x^2 + c/x^3 + d/x^4$, and the corrections arising due to the perturbation are introduced as

$$\bar{x} = x_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \cdots\cdots, \tag{5}$$

where

$$x_0 = -3c \pm \sqrt{9c^2 - 32bd}/4b \tag{6}$$

$$\left.\begin{aligned} &\alpha_1 = -x_0^3/(3c + 4bx_0)\ ;\ \alpha_2 = x_0^5(9c + 10bx_0)/(3c + 4bx_0)^3, \\ &\alpha_3 = -x_0^7(108c^2 + 234bcx_0 + 128b^2x_0^2)/(3c + 4bx_0)^5, \end{aligned}\right\} \tag{7}$$

Using eq. (4) in eq. (2) after changing the independent variable x to $\omega = h(x - \bar{x})$, eq. (2) can be written as

$$\left[\frac{d^2}{d\omega^2} + \frac{\lambda^2}{h^2} - \frac{V(\bar{x})}{h^2} - \frac{V^{(2)}(\bar{x})}{h^4}\frac{1}{2}\omega^2\right]\psi = \sum_{i=3}^{\infty} \frac{V^{(i)}(\bar{x})}{i\,!}\frac{\omega^i}{h^{i+2}}\psi, \tag{8}$$

Now we set

$$h^4 = 2V^{(2)}(\bar{x})\ ;\ \frac{\lambda^2 - V(\bar{x})}{h^2} = \frac{1}{2}q + \frac{\Delta}{h}, \tag{9}$$

and write eq. (8) in the form

$$D_a\psi = \left[\frac{2\Delta}{h} + \sum_{i=3}^{\infty} v_{i2}(\bar{x})\frac{\omega^i}{h^{i-2}}\right]\psi, \tag{10}$$

where

$$D_a = -2\frac{d^2}{d\omega^2} - q + \frac{1}{2}\omega^2$$

$$v_{i2} = \frac{1}{i!}\frac{V^{(i)}(\bar{x})}{V^{(2)}(\bar{x})}$$

$$= \frac{(-1)^{i+1}}{2x_0^{i-2}} \cdot \frac{1}{\beta_0}\left[\delta_0(i) - \delta_1(i)\frac{a}{\beta_0} + \{\beta_1\delta_1(i) - \beta_0\delta_2(i)\}\left(\frac{a}{\beta_0}\right)^2 + \cdots\cdots\right] \tag{11}$$

with

$$\beta_0 = 3\left(b + \frac{2c}{x_0} + \frac{10}{3}\frac{d}{x_0^2}\right)\ ;\ \beta_1 = x_0 - \frac{12\,\alpha_1}{x_0}\left(b + \frac{5c}{2x_0} + \frac{5d}{x_0^2}\right),$$

$$\delta_0(i) = (i+1)\left\{b + (i+2)\frac{c}{2x_0} + (i+2)(i+3)\frac{d}{6x_0^2}\right\},$$

$$\delta_1(i)=(i-2)\Big[bx_0+(i+5)\frac{c}{2}+(i^2+8i+27)\frac{d}{6x_0}$$

$$+3(i+1)\frac{\alpha_1}{x_0}\cdot\Big\{b^2+(i+2)\frac{c^2}{x_0^2}+5(i+2)(i+3)\frac{d^2}{9x_0^4}$$

$$+(i+7)\frac{bc}{2x_0}+(i^2+7i+40)\frac{bd}{6x_0^2}+(i+2)(i+3)\frac{dc}{3x_0^3}\Big\}\Big],$$

$$\delta_2(i)=3(i-2)(i+1)\Big[-\frac{c\alpha_1}{2x_0}-(i+7)\frac{d\alpha_1}{6x_0^2}+\{2\alpha_2x_0-(i+7)\alpha_1^2\}\frac{b^2}{2x_0^3}$$

$$+(i+2)\{2\alpha_2x_0-(i+9)\alpha_1^2\}\frac{c^2}{2x_0^4}+5(i+2)(i+3)\{2\alpha_2x_0-(i+11)\alpha_1^2\}\frac{d^2}{18x_0^6}$$

$$+(i+7)\{2\alpha_2x_0-(i+8)\alpha_1^2\}\frac{bc}{4x_0^3}+(i^2+7i+40)\{2\alpha_2x_0-(i+9)\alpha_1^2\}\frac{db}{12x_0^4}$$

$$+(i+2)(i+9)\{2\alpha_2x_0-(i+10)\alpha_1^2\}\frac{cd}{6x_0^5}\Big]. \tag{12}$$

In lowest order, the r. h. s. of eq. (10) is zero and the solution of

$$D_q\psi_q^{(0)}=0, \tag{13}$$

is $\psi_q^{(0)}(\omega)=D_{\frac{1}{2}(q-1)}(\omega)$, the parabolic cylindrical function. The square integrability of $\psi_q^{(0)}$ demands that q be an odd integer i.e. $q=(2n+1)$, with $n=0, 1, 2, \ldots$. The function $\psi_q^{(0)}(\omega)$ satisfies the recurrence relation

$$\omega\psi_q=(q,\, q+2)\psi_{q+2}+(q,\, q-2)\psi_{q-2}, \tag{14a}$$

where

$$(q,\, q+2)=1, \qquad (q,\, q-2)=\tfrac{1}{2}(q-1). \tag{14b}$$

In general, we write

$$\omega^i\psi_q=\sum_{j=-2i}^{+2i} S_i(q,\, j)\psi_{q+j}, \tag{15}$$

where the coefficients $S_i(q,\, j)$ can be calculated from the repeated use of eq. (14). The lowest order approximation leaves the uncompensated contribution

$$R_q^{(0)}=\Big\{\frac{2\Delta}{h}+\sum_{i=3}^{\infty}v_{i2}(\bar{x})\frac{\omega^i}{h^{i-2}}\Big\}\psi_q(\omega)$$

$$=\sum_{i=3}^{\infty}\frac{1}{h^{i-2}}\sum_{j=-2i}^{+2i}[q,\, q+j]_i\psi_{q+j}(\omega), \tag{16}$$

$$\begin{aligned}&[q,\, q]_3=2\Delta+v_{32}(\bar{x})S_3(q,\, 0) \text{ for } j=0,\\ &[q,\, q+j]_3=v_{32}(\bar{x})S_3(q,\, j) \quad \text{for } j\neq 0,\end{aligned} \tag{17}$$

and

$$\text{for } i>3, \quad -2i \leqslant j \leqslant 2i \quad [q, q+j]_i = v_{i2}(\bar{x}) S_i(q, j). \tag{18}$$

Following the procedure discussed by Muller-Kirsten *et al* (1979), the first order correction to the wave function is given by

$$\psi_a^{(1)} = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=-2i \\ \neq 0}}^{+2i} \frac{[q, q+j]_i}{j} \psi_{a+j}(\omega). \tag{19}$$

The uncompensated term left in first order are

$$R_a^{(1)} = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=-2i \\ \neq 0}}^{+2i} \frac{[q, q+j]_i}{j} R_{a+j}^{(0)}, \tag{20}$$

which yields the next higher order contribution to ψ_a. Similarly, the higher order contributions can be computed and an account of the corresponding uncompensated terms can be maintained. Thus, adding the successive contributions we obtain (Muller-Kirsten et al 1979)

$$\psi_a = \psi_a^{(0)} + \psi_a^{(1)} + \psi_a^{(2)} + \cdots\cdots, \tag{21}$$

which is an asymptotic expansion in powers of $1/h$ and valid in the region $x-\bar{x}=0(1/h)$ i.e. around $x=\bar{x}$. For eq. (21) to be the solution of eq.(10), the sum of the coefficients of ψ_a in $R_a^{(0)}$, $R_a^{(1)}$, $\cdots$-left uncompensated so far must vanish i.e.

$$\begin{aligned} 0 = R_a^{(0)}(j=0) + R_a^{(1)}(j+j'=0) + R_a^{(2)}(j+j'+j''=0) \\ + R_a^{(3)}(j+j'+j''+j'''=0) + \cdots \end{aligned} \tag{22}$$

This is an important result of the present perturbation method as it provides eigenvalues for the corresponding potential.

3. Asymptotic expansion for the energy eigenvalues

In order to get terms up to $1/h^4$ in the eigenvalue expansion of eq. (22), we had to retain terms upto $R_a^{(3)}$. In analogy with eq. (20), we can write for each term in eq. (22) and obtain

$$0 = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} [q, q]_i + \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=-2i \\ \neq 0}}^{+2i} \frac{[q, q+j]_i}{j} \sum_{i'=3}^{8} \frac{1}{h^{i'-2}} [q, q+j]_{i'}$$

$$+\sum_{i=3}^{\infty}\frac{1}{h^{i-2}}\sum_{\substack{j=-2i\\ \neq 0}}^{+2i}\frac{[q,q+j]_i}{j}\sum_{i'=3}^{\infty}\frac{1}{h^{i'-2}}\sum_{\substack{j'=-2i'\\ j+j'\neq 0}}^{+2i'}\frac{[q+j,q+j+j']_{i'}}{j+j'}$$

$$\cdot\sum_{i''=3}^{\infty}\frac{1}{h^{i''-2}}[q+j+j',q]_{i''}$$

$$+\sum_{i=3}^{\infty}\frac{1}{h^{i-2}}\sum_{\substack{j=-2i\\ \neq 0}}^{+2i}\frac{[q,q+j]_i}{j}\sum_{i'=3}^{\infty}\frac{1}{h^{i'-2}}\sum_{\substack{j'=-2i'\\ j+j'\neq 0}}^{+2i'}\frac{[q+j,q+j+j']_{i'}}{j+j'}$$

$$\cdot\sum_{i''=3}^{\infty}\frac{1}{h^{i''-2}}\sum_{\substack{j''=-2i''\\ j+j'+j''\neq 0}}^{+2i''}\frac{[q+j+j',q+j+j'+j'']_{i''}}{j+j'+j''}\sum_{i'''=3}^{\infty}\frac{1}{h^{i'''-2}}$$

$$[q+j+j'+j'',q]_{i'''}+\cdots\cdots\cdots. \tag{23}$$

It may be noted that in this expression a large number of coefficients $[q,q+j]_i$ turn out to be zero. We account here only for the nonvanishing ones, rewrite eq. (23) in descending powers of h and retain the terms upto $1/h^4$ as

$$0=\frac{1}{h}[q,q]_3+\frac{1}{h^2}\Big\{[q,q]_4+\frac{[q,q-6]_3}{-6}[q-6,q]_3+\frac{[q,q-2]_3}{-2}[q-2,q]_3$$

$$+\frac{[q,q+2]_3}{+2}[q+2,q]_3+\frac{[q,q+6]_3}{+6}[q+6,q]_3\Big\}+\frac{1}{h^3}T_3+\frac{1}{h^4}T_4. \tag{24}$$

Here various coefficients are calculated from eq. (18) by making repeated use of the recurrence relation (14a). These coefficients are also obtained using a computer program†. In all previous works, while T_4 and other higher-order terms in eq. (24) are not calculated (as their computation turns out to be lengthy), the T_3 term is also found to be zero. However, this is not the case. The nonvanishing coefficients which contribute to T_3 are

$$T_3=\frac{[q,q-6]_3}{-6}\frac{[q-6,q-6]_3[q-6,q]_3}{-6}+\frac{[q,q-2]_3}{-2}\frac{[q-2,q-2]_3}{-2}$$

$$[q-2,q]_3+\frac{[q,q+2]_3}{+2}\frac{[q+2,q+2]_3[q+2,q]_3}{+2}+\frac{[q,q+6]_3}{+6}$$

$$\frac{[q+6,q+6]_3[q+6,q]_3}{+6} \tag{25a}$$

which after using eqs. (17) and (18) becomes

$$T_3=\frac{1}{36}\Delta.q(41q^2+133)v_{32}^2. \tag{25b}$$

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†I thank Dr Rudolf Muller for extending his help in the computation of these coefficients.

Similarly, T_4 term in eq. (24) can be written as

$$T_4=[q,q]_6+\sum_{\substack{j=-8\\ \neq 0}}^{+8}\frac{[q,q+j]_4[q+j,q]_4}{j}+\sum_{\substack{j=-10\\ \neq 0}}^{+10}\frac{[q,q+j]_5[q+j,q]}{j}$$

$$+\sum_{\substack{j=-6\\ \neq 0}}^{+6}\frac{[q,q+j]_3[q+j,q]_5}{j}+\sum_{\substack{j=-8\\ \neq 0}}^{+8}\frac{[q,q+j]_4}{j}\sum_{j'=-6}^{+6}\frac{[q+j,q+j+j']_3}{j+j'}$$

$$[q+j+j',q]_3+\sum_{\substack{j=-6\\ \neq 0}}^{+6}\frac{[q,q+j]_3}{j}\sum_{\substack{j'=-8\\ j+j'\neq 0}}^{+8}\frac{[q+j,q+j+j']_4[q+j+j',q]_3}{j+j'}$$

$$+\sum_{\substack{j=-6\\ \neq 0}}^{+6}\frac{[q,q+j]_3}{j}\cdot\sum_{\substack{j'=-6\\ j+j'\neq 0}}^{+6}\frac{[q+j,q+j+j']_3[q+j+j',q]_4}{\cdot\ \cdot}$$

$$+\sum_{\substack{j=-6\\ \neq 0}}^{+6}\frac{[q,q+j]_3}{j}\cdot\sum_{\substack{j'=-6\\ j+j'\neq 0}}^{+6}\frac{[q+j,q+j+j']_3}{j+j'}$$

$$\sum_{\substack{j''=-6\\ j+j'+j''\neq 0}}^{+6}\frac{[q+j+j',q+j+j'+j'']_3\,[q+j+j'+j'',q]_3}{j+j'+j''},\qquad (26a)$$

and it further simplifies to the form,

$$T_4=\tfrac{5}{2}q(q^2+5)v_{62}+\frac{1}{2^7}\cdot v_{42}^2\{-(q-1)(q-3)(33q^2-140q+163)$$
$$+(q+1)(q+3)\,.\,(33q^2+140q+163)\}+\frac{5}{2^4.3}v_{52}v_{32}\{-(q-1)$$
$$(19q^3-65q^2+129q-99)+(q+1)(19q^3+65q^2+129q+99)\}$$
$$+\frac{1}{2^7.3}v_{42}v_{32}^2[(q-1)(q-3)(q-5)\,.\,(32q^2-255q+332)$$
$$+6(q-1)(q-2)(q-3)(13q^2-94q+73)+6(q+1)(q+2)(q+3)$$
$$(13q^2+94q+73)+(q+1)(q+3)(q+5)(32q^2+255q+332)$$
$$+18q(q^4+64q^2+55)]+\frac{1}{2^{10}.3^3}\cdot v_{32}^4[-(q-1)(q-3)(q-5)$$
$$(406q^3-4968q^2+17492q-16065)-162(q-1)(q-3)$$
$$(q^2-7q+4)(7q^2-52q+49)+162(q+1)(q+3)\,.\,(q^2+7q+4)$$
$$(7q^2+52q+49)+(q+1)(q+3)(q+5)(406q^3+4968q^2$$
$$+17492q+16065)]\qquad (26b)$$

The energy eigenvalue exansion now becomes

$$0=\frac{2\Delta}{h}+\frac{1}{h^2}[\tfrac{3}{2}(q^2+1)v_{42}+\tfrac{1}{4}(15q^2+7)v_{32}^2]+\frac{1}{h^3}T_3+\frac{1}{h^4}T_4+\Theta\left(\frac{1}{h^5}\right) \quad (27)$$

Besides $1/h$ term in the above expression, $1/h^3$ term also involves Δ which is related to the eigenvalues λ^2 (cf. eq. (9)). Finally, the expansion for energy eigenvalues turns out to be

$$(\lambda^2)_q=\Big[\{V(\bar{x})+\tfrac{1}{2}qh^2\}\{1+\frac{1}{72h^2}q(41q^2+133)v_{32}^2+\cdots\cdots\}$$

$$-\tfrac{1}{2}\{\tfrac{3}{2}(q^2+1)v_{42}+\tfrac{1}{4}(15q^2+7)v_{32}^2\}-\frac{1}{2h^2}T_4+\Theta\left(\frac{1}{h^3}\right)\Big]$$

$$\cdot\ [1+\frac{1}{72h^2}q(41q^2+133)v_{32}^2+\cdots]^{-1}$$

or

$$(\lambda^2)_q=V(\bar{x})+\tfrac{1}{2}qh^2-\tfrac{1}{8}[6(q^2+1)v_{42}+(15q^2+7)v_{32}^2]$$

$$+\frac{1}{2h^2}\Big\{\frac{1}{288}q(41q^2+133)v_{32}^2[6(q^2+1)v_{42}$$

$$+(15q^2+7)v_{32}^2]-T_4\Big\}+\Theta\left(\frac{1}{h^3}\right), \quad (28)$$

where $v_{r2}=v_{r2}(\bar{x})$ can be obtained from eq. (11).

4. Possible extension and generalizations (3-dimensional case)

Various results of Sections 2 and 3 can easily be extended to three-dimensions and to the potential (1). In fact, the radial part of the Schrödinger equation can be written in its reduced form as ($2m=\hbar=1$, $E=\lambda^2$)

$$\left[\frac{d^2}{dr^2}+\lambda^2-\frac{l(l+1)}{r^2}-V(r)\right]\psi(r)=0, \quad (29)$$

where the independent variable r now varies from 0 to ∞. When the potential of eq. (1) is used in eq. (29) the net potential term in the latter equation can be expressed as

$$V(r)=\frac{a}{r^\nu}+\frac{b}{r^{\nu+1}}+\frac{c}{r^{\nu+2}}+\frac{d}{r^{\nu+3}}+\frac{l(l+1)}{r^2}. \quad (30)$$

Two distinct cases arise :

Case I when $l=0$:

For an arbitrary ν, while the asymptotic expansions for energy eigenvalues and eigenfunctions (cf. eqs. (21) and (23)) remain the same in this case also, the computation for $v_{i2}(\bar{r})$ can be carried out. Various expressions now take the form as follow :

$$\bar{r}=r_0+\sum_{i=1}^{\infty}\alpha_i a^i, \tag{31}$$

with

$$r_0=[-(\nu+2)c\pm\sqrt{(\nu+2)^2c^2-4(\nu+1)(\nu+3)bd}]/[2(\nu+1)b],$$

$$\alpha_1=-\nu r_0^3/[(\nu+2)c+2(\nu+1)br_0],$$

$$\alpha_2=\nu^2 r_0^5[3(\nu+2)c+5(\nu+1)br_0]/[(\nu+2)c+2(\nu+1)br_0]^3,$$

$$\alpha_3=\frac{-\nu^3 r_0^7[39(\nu+1)(\nu+2)bcr_0+12(\nu+2)^2c^2+32(\nu+1)^2b^2r_0^2]}{[(\nu+2)c+2(\nu+1)br_0]^5},$$

..................etc.

Further more,

$$v_{i2}(\bar{r})=-\frac{1}{i!}\frac{V^{(i)}(\bar{r})}{V^{(2)}(\bar{r})}$$

$$=\frac{(-1)^{i+1}(\nu+i-1)!}{i!(\nu+1)!r_0^{i-2}}\cdot\frac{1}{\beta_0}\left[\delta_0(i)-\delta_1(i)\frac{a}{\beta}+\{\beta_1\delta_1(i)\right.$$

$$\left.-\beta_0\delta_2(i)\}\left(\frac{a}{\beta_0}\right)^2+\cdots\cdots\right] \tag{32}$$

where

$$\delta_0(i)=\frac{(\nu+i)}{\nu}\left\{b+\frac{(\nu+i+1)}{(\nu+1)}\frac{c}{r_0}+\frac{(\nu+i+1)(\nu+i+2)}{(\nu+1)(\nu+2)}\frac{d}{r_0^2}\right\},$$

$$\delta_1(i)=\frac{(i-2)}{\nu}\left[br_0+\frac{(2\nu+i+3)}{(\nu+1)}c+\left\{\frac{(i^2+i(3\nu+5)+3(\nu+2)^2}{(\nu+1)(\nu+2)}\right\}\frac{d}{r_0}\right.$$

$$+\frac{(\nu+2)(\nu+i)}{\nu}\cdot\frac{\alpha_1}{r_0}\left\{b^2+\frac{(\nu+3)(\nu+i+1)}{(\nu+1)^2}\frac{c^2}{r_0^2}\right.$$

$$+\frac{(\nu+i+1)(\nu+i+2)(\nu+3)(\nu+4)}{(\nu+1)^2(\nu+2)^2}\frac{d^2}{r_0^4}+\frac{(2\nu+i+5)}{(\nu+1)}\frac{bc}{r_0}$$

$$+\frac{[i^2+i(2\nu+5)+2(\nu+2)(\nu+3)]}{(\nu+1)(\nu+2)}\cdot\frac{bd}{r_0^2}$$

$$\left.\left.+\frac{(\nu+i+1)(\nu+3)(2\nu+i+7)}{(\nu+1)^2(\nu+2)}\frac{dc}{r_0^3}\right\}\right],$$

$$\delta_2(i)=\frac{(i-2)(\nu+i)(\nu+2)}{\nu^2}\left[-\frac{\nu}{(\nu+1)}\frac{c\alpha_1}{r_0}-\frac{\nu(2\nu+i+5)}{(\nu+1)(\nu+2)}\cdot\frac{d\alpha_1}{r_0^2}\right.$$

$$+\{2\alpha_2 r_0-(2\nu+i+5)\alpha_1^2\}\frac{b^2}{2r_0^2}+\frac{(\nu+i+1)(\nu+3)}{(\nu+1)^2}\{2\alpha_2 r_0$$

$$-(2\nu+l+7)\alpha_1^2\}\frac{c^2}{2r_0^4}+\frac{(\nu+l+1)(\nu+l+2)(\nu+3)(\nu+4)}{(\nu+1)^2(\nu+2)^2}$$

$$\{2\alpha_2 r_0-(2\nu+l+9)\alpha_1^2\}\frac{d^2}{2r_0^6}+\frac{(2\nu+l+5)}{(\nu+1)}\{2\alpha_2 r_0$$

$$-(2\nu+l+6)\alpha_1^2\}\frac{bc}{2r_0^3}+\frac{[l^2+l(2\nu+5)+2(\nu+3)(\nu+4)]}{(\nu+1)(\nu+2)}$$

$$\{2\alpha_2 r_0-(2\nu+l+7)\alpha_1^2\}\frac{db}{2r_0^4}+\frac{(\nu+l+1)(\nu+3)(2\nu+l+7)}{(\nu+1)^2(\nu+2)^2}$$

$$\{2\alpha_2 r_0-(2\nu+l+8)\alpha_1^2\}\frac{dc}{2r_0^5}\Big].$$

For some specific values of ν, several interesting cases arise from eq. (30) :

(i) $\nu=-2$; $V(r)=ar^2+br+c+d/r$ (both linear and harmonic confining potentials, used in heavy meson spectroscopy)

(ii) $\nu=-1$; $V(r)=ar+b+c/r+d/r^2$ (linear confinement potential used in baryonium case)

(iii) $\nu=0$; $V(r)=a+b/r+c/r^2+d/r^3$ - -

(iv) $\nu=+1$; $V(r)=a/r+b/r^2+c/r^3+d/r^4$ (used in superpositronium case)

Case II : when $l \neq 0$:

For an arbitrary ν, one can still follow the procedure of Sections 2 and 3 and rederive $v_{l2}(\bar{r})$ in this case also. Sometimes the change of independent variable r to $z=\ln r$ is also helpful (Muller-Kirsten *et al* 1979) since in this case the domain of the independent variable z becomes $-\infty < z < +\infty$ as for x in Section. 2. However, for some choices of ν (say $\nu=1, 2$ or $\nu=0 (r \neq 0)$) the last term in eq. (30) can easily be combined with one of the terms in $V(r)$.

5. Discussion and summary

Using the perturbation method of Muller-Kirsten and his coworkers we have obtained the improved asymptotic expansion (cf. eq. (28)) for the eigenenergies upto order $1/h^4$. As such, the results are quite general and valid for any arbitrary potential. However, the computation of the various terms in the expansion has been carried out for the potential (3) (and for that matter for the potential (1)) by assuming a/x-term as small.

For given potential parameters though it is trivial to compute the energy eigenvalues and eigenfunctions from the expansions (28) and (21) respectively, we have not obtained numerical estimates of various terms. This is mainly because in an asymptotic expansion the successive contributions may not be a decreasing sequence, rather they may vary randomly. In such cases, one has to adopt certain well defined criteria (e.g. Dingle's convergence factor (Dingle 1973)) to obtain an asymptotically correct value of the sum by terminating the power series. Hence, the relative contribution of various terms in the expansion do not alter the conclusions.

One can easily obtain the zero energy solutions of the Schrödinger equation using the newly developed tools of super-symmetric quantum mechanics (Cooper and Freedman 1983, Kostelecky and Nieto 1984, Khare 1985) for the potential (1) with $\nu=1$ and for both the cases $a=0$ and $a \neq 0$. If we choose the superpotential $W(r)$ as

$$W(r)=\alpha+\frac{\beta}{r}+\frac{\gamma}{r^2} \tag{33}$$

then the corresponding potential in eq. (29) becomes

$$V(r)=W^2-W'=\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\frac{d}{r^4}+\alpha^2 \tag{34}$$

with

$$a=2\alpha\beta,\ b=\beta+2\alpha\gamma,\ c=2\gamma+2\beta\gamma,\ d=\gamma^2.$$

The normalizable ground state wave function ψ_0 corresponding to zero-energy eigenvalue can be found from the expression (Kostelecky and Nieto 1984)

$$\psi_0=\exp\left[-\int W(r)\,dr\right]$$

as

$$\psi_0=r^{|\beta|}e^{-\alpha r-\gamma/r} \tag{35}$$

with

$$\alpha>0,\ |\beta|>1,\ \gamma>0.$$

Obviously, in this case, one has to redefine the potential as $V(r)-\alpha^2$. For $a=0$ case, we put $\beta=0$ in eq. (35) and obtain the solution

$$\psi_0=e^{-\alpha r-\gamma/r}$$

corresponding to the potential (34) with $a=0$.

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